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New Matrix Lie Algebra and Its Application

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Abstract A set of new matrix Lie algebra and its corresponding loop algebra are constructed. By making use of Tu scheme, a Liouville integrable multi-component hierarchy of soliton equation is generated. As its reduction cases, the multi-component Tu hierarchy is given. Finally, the multi-component integrable coupling system of Tu hierarchy is presented through enlarging matrix spectral problem.

Keywords Matrix Lie algebra · Liouville integrable system · Hamiltonian structure · Integrable couplings

1 Introduction

Professor Tu once proposed an efficient method for obtaining integrable Hamiltonian hierarchies of soliton equation with infinite dimensions in [1]. Professor Ma [2] further developed the method and called it Tu model. By use of Tu model, some well-known integrable Hamiltonian hierarchies were worked out, such as AKNS hierarchy, KN hierarchy, TC hierarchy, BPT hierarchy, etc. [3–15]. With the development of soliton theory, the integrable coupling is quite a new and significant subject, which originates from the investigation of centerless Virasoro symmetry algebras of soliton equations. For a given integrable hierarchy $U_t = K(u)$,we can construct a new bigger triangular integrable system as follows:

$$\begin{cases} U_t = K(u), \\ V_t = S(u, v). \end{cases} \tag{1}$$

This is usually called the integrable couplings of $U_t = K(u)$ if the system (1) is still an integrable system. In this paper, we firstly construct a new multi-component matrix Lie algebra, and denote it as A_{2M}. Then it follows that a corresponding loop algebra $\tilde{A}_{2M}$ is

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constructed. By use of Tu model, a multi-component integrable hierarchy is given, which is Liouville integrable and possesses Hamiltonian structure. Finally, with the help of expanding matrix Lie algebra, the integrable coupling of the above system is produced.

2 A New Matrix Lie Algebra and Its Application

First, we construct a multi-component matrix Lie algebra:

$$\begin{cases} e_1 = \frac{1}{2}\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}_{2M\times 2M}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}_{2M\times 2M}, \\ e_3 = \frac{1}{2}\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}_{2M\times 2M}, \quad E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{M\times M}, \\ [e_1, e_2] = e_3, \quad [e_1, e_3] = e_2, \quad [e_3, e_2] = e_1. \end{cases} \tag{2}$$

Definition 1 Set

$$W = (w_{ij})_{M\times M}, \tag{3}$$

$$P = \begin{pmatrix} P^{(1)} & P^{(2)} \\ P^{(3)} & P^{(4)} \end{pmatrix}_{2M\times 2M}, \tag{4}$$

$p^{(n)} = (p_{ij}^{(n)})_{M\times M}, n = 1, \ldots, 4$. We define the product of matrix W and P as follows:

$$WP = \begin{pmatrix} WP^{(1)} & WP^{(2)} \\ WP^{(3)} & WP^{(4)} \end{pmatrix}_{2M\times 2M}. \tag{5}$$

In terms of (2), a loop algebra $\tilde{A}_{2M}$ is presented as

$$\begin{cases} e_i(n) = e_i\lambda^n, \\ [e_1(m), e_2(n)] = e_3(m+n), \\ [e_1(m), e_3(n)] = e_2(m+n), \\ [e_3(m), e_2(n)] = e_1(m+n), \\ \deg(e_i(n)) = n, \quad i = 1, 2, 3. \end{cases} \tag{6}$$

Consider an isospectral problem as follows

$$\begin{cases} \varphi_x = U\varphi, \quad \lambda_t = 0, \quad U = -2e_1(1) + Qe_1(0) + 2Re_2(0), \\ Q = (Q_{ij})_{M\times M}, \quad Q_{ij} = q_i\delta_{ij}, \quad R = (R_{ij})_{M\times M}, \quad R_{ij} = r_i\delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \end{cases} \tag{7}$$

and set

$$\begin{cases} V = \sum_{m=0}^{\infty}(A_m e_1(-m) + B_m e_2(-m) + C_m e_3(-m)), \\ A_m = (A_{ij})_{M\times M}, \quad A_{ij} = a_i^{(m)}\delta_{ij}, \\ B_m = (B_{ij})_{M\times M}, \quad B_{ij} = b_i^{(m)}\delta_{ij}, \\ C_m = (C_{ij})_{M\times M}, \quad C_{ij} = c_i^{(m)}\delta_{ij}. \end{cases} \tag{8}$$

Solving the stationary zero curvature equation

$$V_x = [U, V] \tag{9}$$

gives rise to

$$\begin{cases} A_{mx} = -2RC_m, \\ B_{mx} = QC_m - 2C_{m+1}, \\ C_{mx} = QB_m - 2RA_m - 2B_{m+1}, \\ A_0 = E, \quad B_0 = C_0 = 0, \quad A_1 = 0, \quad B_1 = -R, \quad C_1 = 0. \end{cases} \tag{10}$$

Set

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^{n}(A_m e_1(n-m) + B_m e_3(n-m) + C_m e_4(n-m)), \\ V_-^{(n)} &= V\lambda^n - V_+^{(n)}, \end{aligned} \tag{11}$$

then (9) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{12}$$

We find that the terms on the left-hand side in (12) are of degree ≥ 0, while the terms on the right-hand side in (12) are of degree ≤ 0. Therefore,

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = 2C_{n+1}e_2(0) + 2B_{n+1}e_3(0). \tag{13}$$

Taking $V^{(n)} = V_+^{(n)} + \Delta_n$, $\Delta_n = R^{-1}B_{n+1}e_1(0)$, then the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{14}$$

leads to

$$u_t = \begin{pmatrix} Q \\ R \end{pmatrix}_{2M\times Mt} = \begin{pmatrix} \partial R^{-1}B_{n+1} \\ \frac{1}{2}R^{-1}\partial A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \partial R^{-1} \\ R^{-1}\partial & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \end{pmatrix} = J\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \end{pmatrix}. \tag{15}$$

From (10), the recurrence operator L satisfy

$$\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \end{pmatrix}_{2M\times M} = L\begin{pmatrix} \frac{1}{2}A_n \\ B_n \end{pmatrix}_{2M\times M} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}_{2M\times 2M}\begin{pmatrix} \frac{1}{2}A_n \\ B_n \end{pmatrix}_{2M\times M}, \tag{16}$$

with

$$\begin{aligned} L_{11} &= \left(\frac{1}{2}\partial^{-1}q_i\partial\delta_{ij}\right)_{M\times M}, \quad L_{12} = \left(2\partial^{-1}\, r_i\partial\delta_{ij}\right)_{M\times M}, \\ L_{21} &= \left(\frac{1}{2}\partial\frac{1}{r_i}\partial\delta_{ij}\right)_{M\times M}, \quad L_{22} = \left(\frac{1}{2}\, q_i\delta_{ij}\right)_{M\times M}, \quad \partial^{-1}\partial = \partial\partial^{-1} = 1. \end{aligned} \tag{17}$$

Thus, the system (15) can be written as

$$U_t = \begin{pmatrix} Q \\ R \end{pmatrix}_{2M\times Mt} = JL^n\begin{pmatrix} 0 \\ -R \end{pmatrix}_{2M\times M}. \tag{18}$$

If we taking $M = 1$, the system (18) reduces to the Tu hierarchy, therefore we conclude that its the multi-component integrable system of the Tu hierarchy.

3 The Hamiltonian Structure of the System (15)

Denote

$$V = Ae_1(0) + Be_2(0) + Ce_3(0) = \frac{1}{2}\begin{pmatrix} A & B+C \\ B-C & -A \end{pmatrix}. \tag{19}$$

A direct calculation gives

$$\left\langle V, \frac{\partial U}{\partial Q}\right\rangle = \frac{1}{2}\,A, \qquad \left\langle V, \frac{\partial U}{\partial R}\right\rangle = B, \qquad \left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = -A. \tag{20}$$

Substitute (20) into trace identity leads to

$$\begin{pmatrix} \frac{\delta}{\delta Q} \\ \frac{\delta}{\delta R} \end{pmatrix}(-A) = \lambda^{-\gamma}\left(\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} \frac{1}{2}A \\ B \end{pmatrix}\right). \tag{21}$$

Comparing the coefficient of the λ^{-n-1} yields

$$\begin{pmatrix} \frac{\delta}{\delta Q} \\ \frac{\delta}{\delta R} \end{pmatrix}(-A_{n+1}) = (\gamma - n)\begin{pmatrix} \frac{1}{2}A_n \\ B_n \end{pmatrix}. \tag{22}$$

Let $n = 0$ in (22) and find that $\gamma = 0$, so

$$\frac{\delta H_n}{\delta u} = \begin{pmatrix} \frac{1}{2}A_n \\ B_n \end{pmatrix}, \qquad H_n = \frac{1}{n}A_{n+1}. \tag{23}$$

Hence, we obtain multi-component Hamiltonian structure of multi-component Tu hierarchy:

- when $n = 1$, $H_1 = (H_{1i})_{M\times M}$, $H_{1i} = -\frac{1}{2}\,r_i^2$,
- when $n = 2$, $H_2 = (H_{2i})_{M\times M}$, $H_{2i} = -\frac{1}{4}\,r_i^2 q_i$.

Thus, the Hamiltonian structure of the system is given by

$$u_t = J\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \qquad H_{n+1} = \frac{1}{n+1}A_{n+2}. \tag{24}$$

It is easy to verify that $JL = L^*J$, therefore the system (18) is Liouville integrable.

4 Integrable Couplings of Multi-Component Tu Hierarchy

We construct expanding matrix Lie algebra of (2)

$$\begin{cases} \bar{e}_1=\begin{pmatrix} e_1 & 0\\ 0 & e_1\end{pmatrix}_{4M\times 4M}, & \bar{e}_2=\begin{pmatrix} e_2 & 0\\ 0 & e_2\end{pmatrix}_{4M\times 4M}, \qquad \bar{e}_3=\begin{pmatrix} e_3 & 0\\ 0 & e_3\end{pmatrix}_{4M\times 4M},\\ \bar{e}_4=\begin{pmatrix} 0 & e_1\\ 0 & 0\end{pmatrix}_{4M\times 4M}, & \bar{e}_5=\begin{pmatrix} 0 & e_2\\ 0 & 0\end{pmatrix}_{4M\times 4M}, \qquad \bar{e}_6=\begin{pmatrix} 0 & e_3\\ 0 & 0\end{pmatrix}_{4M\times 4M},\\ e_1=\frac{1}{2}\begin{pmatrix} E & 0\\ 0 & -E\end{pmatrix}_{2M\times 2M}, & e_2=\frac{1}{2}\begin{pmatrix} 0 & E\\ E & 0\end{pmatrix}_{2M\times 2M},\\ e_3=\frac{1}{2}\begin{pmatrix} 0 & E\\ -E & 0\end{pmatrix}_{2M\times 2M}, & E=\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}_{M\times M},\\ [\bar{e}_1,\bar{e}_2]=\bar{e}_3,\quad [\bar{e}_1,\bar{e}_3]=\bar{e}_2,\quad [\bar{e}_3,\bar{e}_2]=\bar{e}_1,\quad [\bar{e}_1,\bar{e}_5]=\bar{e}_6,\quad [\bar{e}_1,\bar{e}_6]=\bar{e}_5,\\ [\bar{e}_2,\bar{e}_4]=-\bar{e}_6,\quad [\bar{e}_2,\bar{e}_6]=-\bar{e}_4,\quad [\bar{e}_3,\bar{e}_4]=-\bar{e}_5,\quad [\bar{e}_3,\bar{e}_5]=\bar{e}_4,\\ [\bar{e}_1,\bar{e}_4]=[\bar{e}_2,\bar{e}_5]=[\bar{e}_3,\bar{e}_2]=[\bar{e}_3,\bar{e}_6]=[\bar{e}_4,\bar{e}_5]=[\bar{e}_4,\bar{e}_6]=[\bar{e}_5,\bar{e}_2]=[\bar{e}_5,\bar{e}_6]=0. \end{cases} \tag{25}$$

Definition 2 Set

$$W=(w_{ij})_{M\times M}, \tag{26}$$

$$K=\begin{pmatrix} K^{(1)} & K^{(2)} & K^{(3)} & K^{(4)}\\ K^{(5)} & K^{(6)} & K^{(7)} & K^{(8)}\\ K^{(9)} & K^{(10)} & K^{(11)} & K^{(12)}\\ K^{(13)} & K^{(14)} & K^{(15)} & K^{(16)}\end{pmatrix}_{4M\times 4M}, \tag{27}$$

$K^{(n)}=(K_{ij}^{(n)})_{M\times M},\ n=1,\ldots,16$. We define the product of matrix W and K as follows:

$$WK=\begin{pmatrix} WK^{(1)} & WK^{(2)} & WK^{(3)} & WK^{(4)}\\ WK^{(5)} & WK^{(6)} & WK^{(7)} & WK^{(8)}\\ WK^{(9)} & WK^{(10)} & WK^{(11)} & WK^{(12)}\\ WK^{(13)} & WK^{(14)} & WK^{(15)} & WK^{(16)}\end{pmatrix}_{4M\times 4M}. \tag{28}$$

Then the corresponding loop algebra is given as follows:

$$\begin{cases} \bar{e}_i(n)=\bar{e}_i\lambda^n,\quad [\bar{e}_1(m),\bar{e}_2(n)]=\bar{e}_3(m+n),\quad [\bar{e}_1(m),\bar{e}_3(n)]=\bar{e}_2(m+n),\\ [\bar{e}_3(m),\bar{e}_2(n)]=\bar{e}_1(m+n),\quad [\bar{e}_1(m),\bar{e}_5(n)]=\bar{e}_6(m+n),\quad [\bar{e}_1(m),\bar{e}_6(n)]=\bar{e}_5(m+n),\\ [\bar{e}_2(m),\bar{e}_4(n)]=-\bar{e}_6(m+n),\quad [\bar{e}_2(m),\bar{e}_6(n)]=-\bar{e}_4(m+n),\\ [\bar{e}_3(m),\bar{e}_4(n)]=-\bar{e}_5(m+n),\quad [\bar{e}_3(m),\bar{e}_5(n)]=\bar{e}_4(m+n),\\ [\bar{e}_1(m),\bar{e}_4(n)]=[\bar{e}_2(m),\bar{e}_5(n)]=[\bar{e}_3(m),\bar{e}_2(n)]=[\bar{e}_3(m),\bar{e}_6(n)]=[\bar{e}_4(m),\bar{e}_5(n)]\\ \quad=[\bar{e}_4(m),\bar{e}_6(n)]=[\bar{e}_5(m),\bar{e}_2(n)]=[\bar{e}_5(m),\bar{e}_6(n)]=0,\\ \deg(e_i(n))=n,\quad i=1,2,\ldots,6. \end{cases} \tag{29}$$

Consider an isospectral problem as follows

$$\begin{cases} \varphi_x = U\varphi, \quad \lambda_t = 0, \quad U = -2\bar{e}_1(1) + Q\bar{e}_1(0) + 2R\bar{e}_2(0) + S\bar{e}_4(0) + 2T\bar{e}_5(0), \\ Q = (Q_{ij})_{M\times M}, \quad Q_{ij} = q_i\delta_{ij}, \quad R = (R_{ij})_{M\times M}, \quad R_{ij} = r_i\delta_{ij}, \\ S = (S_{ij})_{M\times M}, \quad S_{ij} = s_i\delta_{ij}, \quad T = (T_{ij})_{M\times M}, \quad T_{ij} = t_i\delta_{ij}. \end{cases} \tag{30}$$

Set

$$\begin{cases} V = \sum_{m=0}^{\infty}(A_m\bar{e}_1(-m) + B_m\bar{e}_2(-m) + C_m\bar{e}_3(-m) + D_m\bar{e}_4(-m) \\ \qquad + F_m\bar{e}_5(-m) + G_m\bar{e}_6(-m)), \\ A_m = (A_{ij})_{M\times M}, \quad A_{ij} = a_i^{(m)}\delta_{ij}, \quad B_m = (B_{ij})_{M\times M}, \quad B_{ij} = b_i^{(m)}\delta_{ij}, \\ C_m = (C_{ij})_{M\times M}, \quad C_{ij} = c_i^{(m)}\delta_{ij}, \quad D_m = (D_{ij})_{M\times M}, \quad D_{ij} = d_i^{(m)}\delta_{ij}, \\ F_m = (F_{ij})_{M\times M}, \quad F_{ij} = f_i^{(m)}\delta_{ij}, \quad G_m = (G_{ij})_{M\times M}, \quad G_{ij} = g_i^{(m)}\delta_{ij}. \end{cases} \tag{31}$$

Solving the stationary zero curvature equation

$$V_x = [U, V] \tag{32}$$

gives rise to

$$\begin{cases} A_{mx} = -2RC_m, \\ B_{mx} = QC_m - 2C_{m+1}, \\ C_{mx} = QB_m - 2B_{m+1} - 2RA_m, \\ D_{mx} = -2RG_m - 2TC_m, \\ F_{mx} = -2G_{m+1} + QG_m + SC_m, \\ G_{mx} = -2F_{m+1} + QF_m - 2RD_m + SB_m - 2TA_m, \\ A_0 = E, \quad B_0 = C_0 = D_0 = F_0 = G_0 = 0, \\ B_1 = -R, \quad F_1 = -T, \quad A_1 = C_1 = D_1 = G_1 = 0. \end{cases} \tag{33}$$

Set

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^{n}(A_m\bar{e}_1(n-m) + B_m\bar{e}_2(n-m) + C_m\bar{e}_3(n-m)) + D_m\bar{e}_4(n-m) \\ &\quad + F_m\bar{e}_5(n-m) + G_m\bar{e}_6(n-m)), \quad V_-^{(n)} = V\lambda^n - V_+^{(n)}, \end{aligned} \tag{34}$$

then directly calculation yields

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = 2C_{n+1}\bar{e}_2(0) + 2B_{n+1}\bar{e}_3(0) + 2G_{n+1}\bar{e}_5(0) + 2F_{n+1}\bar{e}_6(0). \tag{35}$$

Taking $V^{(n)} = V_+^{(n)} + \Delta_n$, $\Delta_n = R^{-1}B_{n+1}\bar{e}_1(0) + (-(R^{-1})^2TB_{n+1} + R^{-1}F_{n+1})\bar{e}_4(0)$ then gives

$$u_t = \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix}_{4M\times Mt} = \begin{pmatrix} \partial R^{-1}B_{n+1} \\ -C_{n+1} \\ -\partial(R^{-1})^2TB_{n+1} + \partial R^{-1}F_{n+1} \\ -G_{n+1} \end{pmatrix}_{4M\times M}$$

$$=\begin{pmatrix} 0 & \partial R^{-1} & 0 & 0 \\ R^{-1}\partial & 0 & 0 & 0 \\ 0 & -\partial (R^{-1})^2 T & \partial R^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \\ F_{n+1} \\ G_{n+1} \end{pmatrix} = J\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \\ F_{n+1} \\ G_{n+1} \end{pmatrix}. \tag{36}$$

From (33), the recurrence operator L satisfy

$$\begin{pmatrix} \frac{1}{2}A_{n+1} \\ B_{n+1} \\ F_{n+1} \\ G_{n+1} \end{pmatrix} = L\begin{pmatrix} \frac{1}{2}A_{n} \\ B_{n} \\ F_{n} \\ G_{n} \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & 0 & L_{43} & L_{44} \end{pmatrix}\begin{pmatrix} \frac{1}{2}A_{n} \\ B_{n} \\ F_{n} \\ G_{n} \end{pmatrix}, \tag{37}$$

where

$$L_{22} = L_{33} = L_{44} = \left(-\frac{1}{2}q_i\delta_{ij}\right)_{M\times M}, \qquad L_{11} = \left(\left(-\frac{1}{2}\partial^{-1}q_i\partial\right)\delta_{ij}\right)_{M\times M},$$

$$L_{12} = \left(\left(2\partial^{-1}r_i\partial\right)\delta_{ij}\right)_{M\times M}, \qquad L_{21} = \left(\left(\frac{1}{2}\partial\frac{1}{r_i}\partial - 2r_i\right)\delta_{ij}\right)_{M\times M},$$

$$L_{31} = \left(\left(-2t_i - 2r_i\partial^{-1}t_i\frac{1}{r_i}\partial\right)\delta_{ij}\right)_{M\times M}, \qquad L_{32} = \left(\frac{1}{2}s_i\delta_{ij}\right)_{M\times M},$$

$$L_{34} = \left(\left(-\frac{1}{2}\partial + 2r_i\partial^{-1}r_i\right)\delta_{ij}\right)_{M\times M}, \qquad L_{41} = \left(\left(-\frac{1}{2}s_i\frac{1}{r_i}\partial\right)\delta_{ij}\right)_{M\times M},$$

$$L_{43} = \left(\left(-\frac{1}{2}\partial\right)\delta_{ij}\right)_{M\times M}, \qquad L_{44} = \left(\left(\frac{1}{2}q_i\right)\delta_{ij}\right)_{M\times M}.$$

Therefore, the system (36) can be written as

$$U_t = \begin{pmatrix} Q \\ R \\ S \\ T \end{pmatrix}_t = JL^n\begin{pmatrix} \frac{1}{2}A_1 \\ B_1 \\ F_1 \\ G_1 \end{pmatrix} = JL^n\begin{pmatrix} 0 \\ -R \\ -T \\ 0 \end{pmatrix}. \tag{38}$$

When taking $M = 1$ in (38), the system (38) is the integrable coupling system of the Tu hierarchy. When taking $M > 1$, the system is the multi-component integrable couplings system of the Tu hierarchy.

5 Conclusion

In this paper, we have obtained a multi-component Liouville integrable hierarchy by use of the loop algebra (2). In a similar way to this paper we can get other multi-component integrable system, we will discuss these problems in the future.

References

1. Tu, G.Z.: J. Math. Phys **30**, 330 (1989)
2. Ma, W.X.: Chin. J. Contemp. Math **13**, 79 (1992)
3. Zhang, Y.F., Yan, Q.Y.: Acta Phys. Sin. **52**, 2109 (2003) (in Chinese)

4. Dong, H.H., Zhang, N.: Chinese Phys. **15**, 1919 (2006)
5. Tu, G.Z., Ma, W.X.: J. Partial Differ. Equ. **3**, 53 (1992)
6. Xia, T.C.: Acta Math. Phys. **19**, 507 (1999)
7. Ma, W.X.: J. Phys. A **25**, 719 (1992)
8. Ma, W.X., Zhou, R.G.: Chin. Ann. Math. B **23**, 373 (2002)
9. Guo, F.K., Zhang, Y.F.: Acta Phys. Sin **51**, 951 (2002) (in Chinese)
10. Li, Z., Dong, H.: Mod. Phys. Lett. B **21**, 407–413 (2007)
11. Li, Z., Zhang, Y., Dong, H.: Mod. Phys. Lett. B **21**, 595–602 (2007)
12. Guo, F.K.: Acta Math. Appl. Sin. **23**, 181 (2000)
13. Fan, E.G.: Physica A **301**, 105 (2001)
14. Zhang, Y.F., Tan, H.W.: Chinese Phys. **13**, 1 (2004)
15. Dong, H.H., Zhang, N.: Commun. Theor. Phys **44**, 997 (2005)